# A Comparative Analysis of the Differential Transform Method and the Adomian Decomposition Method for Solving the Burgers Equation

Ahmad M. D. Al-Eybani*

*(The Public Authority for Applied Education and Training (PAAET), Kuwait)

**DOI: https://doi.org/10.5281/zenodo.15303937**

**Published Date: 29-April-2025**

***Abstract:*** **The Burgers equation, a fundamental partial differential equation (PDE) in applied mathematics and physics, is widely studied due to its applications in fluid dynamics, gas dynamics, traffic flow, and nonlinear wave propagation. Named after Johannes Martinus Burgers, this equation combines nonlinear convection and diffusion, making it a simplified model for complex phenomena like turbulence and shock waves. The one-dimensional Burgers equation is typically expressed as:**

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$

**where $u(x,t)$ represents the velocity field, $\nu > 0$ is the kinematic viscosity, $t$ is time, and $x$ is the spatial coordinate. The nonlinear term $u\frac{\partial u}{\partial x}$ introduces complexity, making analytical solutions challenging except in specific cases. As a result, semi-analytical and numerical methods have been developed to approximate solutions to the Burgers equation.**

**Among these methods, the Differential Transform Method (DTM) and the Adomian Decomposition Method (ADM) are two powerful semi-analytical techniques that have gained attention for their ability to handle nonlinear PDEs efficiently. This article provides a detailed comparison of DTM and ADM when applied to the Burgers equation, examining their theoretical foundations, implementation procedures, advantages, limitations, and performance in terms of accuracy, computational efficiency, and applicability.**

***Keywords:*** **Differential Transform Method (DTM), Adomian Decomposition Method (ADM), Burgers equation.**

## 1. OVERVIEW OF THE DIFFERENTIAL TRANSFORM METHOD (DTM)

The Differential Transform Method is a semi-analytical technique based on Taylor series expansions. It transforms a differential equation and its initial or boundary conditions into an algebraic equation in the transform domain, which can be solved iteratively. The DTM was first introduced by Zhou in 1986 for solving linear and nonlinear electrical circuit problems and has since been extended to various differential equations, including PDEs like the Burgers equation.

### 1.1 Theoretical Basis of DTM

The DTM is grounded in the concept of the differential transform, which is defined for a function $u(x,t)$ at a point $(x_0, t_0)$ as follows:

$$U(k,h) = \frac{1}{k!\,h!}\left[\frac{\partial^{k+h}}{\partial x^k \partial t^h} u(x,t)\right]_{\substack{x=x_0\\ t=t_0}}$$

where $U(k,h)$ is the differential transform of $u(x,t)$, and $k$ and $h$ are the transform variables corresponding to the spatial and temporal derivatives, respectively. The inverse transform reconstructs the solution as:

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k,h)(x-x_0)^k(t-t_0)^h$$

For practical computations, the series is truncated to a finite number of terms, yielding an approximate solution. The DTM converts the Burgers equation into a recurrence relation by applying transform rules for derivatives and nonlinear terms.

**1.2 Application of DTM to the Burgers Equation**

To apply DTM to the Burgers equation, consider the standard form:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$

with initial condition $u(x,0) = f(x)$

and appropriate boundary conditions. The DTM transforms each term as follows:

- **Time derivative:** $\frac{\partial u}{\partial t} \rightarrow (h+1)U(k,h+1)$
- **Spatial second derivative:** $\frac{\partial^2 u}{\partial x^2} \rightarrow (k+1)(k+2)U(k+2,h)$
- **Nonlinear term:** The term $u\frac{\partial u}{\partial x}$ is handled using the convolution property of DTM. If $\frac{\partial u}{\partial x}$ , then $V(k,h) = (k+1)U(k+1,h)$, and the transform of the product is:

$$U.V = \sum_{m=0}^{k}\sum_{n=0}^{h} U(m,n)V(k-m,h-n)$$

Applying these transforms to the Burgers equation yields a recurrence relation for $U(k,h)$:

$$(h+1)U(k,h+1) = \nu(k+1)(k+2)U(k+2,h) - \sum_{m=0}^{k}\sum_{n=0}^{h} U(m,n)V(k-m,h-n)$$

The initial condition $u(x,0) = f(x)$

is transformed to provide $U(k,0)$, and the recurrence relation is solved iteratively to compute $U(k,h)$ for higher $k$ and $h$. The approximate solution is then obtained via the inverse transform.

**1.3 Advantages of DTM**

Simplicity: DTM transforms PDEs into algebraic equations, which are easier to solve than differential equations.

Accuracy: For smooth solutions, DTM provides highly accurate results with relatively few terms due to its Taylor series basis.

Flexibility: It can handle both initial and boundary value problems and is applicable to linear and nonlinear equations.

No Discretization: Unlike numerical methods like finite difference or finite element methods, DTM does not require spatial or temporal discretization.

**1.4 Limitations of DTM**

Convergence Issues: The Taylor series may have a limited radius of convergence, especially for problems with singularities or sharp gradients.

- **Computational Cost:** Computing higher-order terms for the nonlinear term involves complex convolutions, increasing computational effort.
- **Truncation Errors:** The accuracy depends on the number of terms used, and truncation may lead to errors for long-time simulations.

## 2. OVERVIEW OF THE ADOMIAN DECOMPOSITION METHOD (ADM)

The Adomian Decomposition Method, developed by George Adomian in the 1980s, is a semi-analytical technique for solving linear and nonlinear differential equations. ADM decomposes the solution into an infinite series and handles nonlinear terms using Adomian polynomials, which represent the nonlinearities in a systematic way. The method has been widely applied to PDEs, including the Burgers equation, due to its ability to provide rapidly converging series solutions.

### 2.1 Theoretical Basis of ADM

ADM assumes that the solution to a differential equation can be expressed as an infinite series:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$

The differential equation is written in the form:

$Lu + Ru + Nu = g,$

where $L$ is the highest-order linear differential operator, $R$ is the remaining linear operator, $N$ represents the nonlinear term, and $g$ is the source term. For the Burgers equation, there is no source term $g = 0$, and the nonlinear term is $u\frac{\partial u}{\partial x}$.

The inverse operator $L^{-1}$ is applied to both sides, and the nonlinear term is expanded using Adomian polynomials $A_n$, which are defined for a nonlinearity $Nu = f(u)$ as:

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[f\left(\sum_{k=0}^{\infty} \lambda^k u_k\right)\right]_{\lambda=0}$$

The solution components $u_n$ are computed recursively, starting with an initial guess $u_0$, often based on the initial condition.

### 2.2 Application of ADM to the Burgers Equation

For the Burgers equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$

we define $L = \frac{\partial}{\partial t}$, so the inverse operator is $L^{-1} = \int_0^t . \, dt$ . The equation is rewritten as:

$$Lu = -u\frac{\partial u}{\partial x} + \nu\frac{\partial^2 u}{\partial x^2}$$

Applying $L^{-1}$:

$$u(x,t) = u(x,0) + L^{-1}(\nu\frac{\partial^2 u}{\partial x^2} - u\frac{\partial u}{\partial x})$$

The solution is $u = \sum_{n=0}^{\infty} u_n$ , and the nonlinear term $u\frac{\partial u}{\partial x}$ is represented by Adomian polynomials $A_n$ . The recursive scheme is:

$$u_0 = u(x,0)$$

$$u_{n+1} = L^{-1}\left(\nu\frac{\partial^2 u_n}{\partial x^2} - A_n\right), \qquad n \geq 0$$

The Adomian polynomials for $Nu = u\frac{\partial u}{\partial x}$ are computed as:

$$A_0 = u_0\frac{\partial u_0}{\partial x}$$

$$A_1 = u_0\frac{\partial u_1}{\partial x} + u_1\frac{\partial u_0}{\partial x}$$

$$A_2 = u_0 \frac{\partial u_2}{\partial x} + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_0}{\partial x}$$

and so on. The solution is approximated by truncating the series:

$$u \approx \sum_{n=0}^{N} u_n$$

**2.3 Advantages of ADM**

- **Rapid Convergence:** For many problems, ADM provides accurate solutions with only a few terms.
- **Analytical Nature:** The method yields a series solution that retains the analytical structure of the problem.
- **Handling Nonlinearity:** Adomian polynomials systematically handle complex nonlinear terms.
- **No Discretization**: Like DTM, ADM does not require mesh generation or numerical discretization.

**2.4 Limitations of ADM**

- **Complexity of Polynomials:** Computing Adomian polynomials for high-order terms can be computationally intensive, especially for strongly nonlinear problems.
- **Convergence Dependence:** The convergence of the series depends on the initial condition and the nature of the nonlinearity, and it may fail for problems with discontinuities.
- **Long-Time Behavior:** The series solution may diverge for large $t$, requiring modifications like the multistage ADM.

## 3. COMPARISON OF DTM AND ADM FOR THE BURGERS EQUATION

To compare DTM and ADM, we evaluate their performance based on several criteria: ease of implementation, accuracy, computational efficiency, handling of nonlinearity, and applicability to different forms of the Burgers equation.

**3.1 Ease of Implementation**

- **DTM:** DTM is relatively straightforward to implement, as it involves transforming the PDE into a recurrence relation using predefined transform rules. However, computing the transform of the nonlinear term requires careful handling of convolutions, which can be tedious for high-order terms.
- **ADM:** ADM requires the computation of Adomian polynomials, which can be more complex than DTM's convolutions, especially for intricate nonlinearities. However, once the polynomials are derived, the recursive scheme is systematic and easy to code.

**Verdict:** DTM is slightly easier to implement for beginners due to its reliance on Taylor series, but both methods require familiarity with their respective frameworks.

**3.2 Accuracy**

- **DTM:** DTM's accuracy depends on the number of terms in the Taylor series. For smooth solutions ,DTM provides highly accurate results. However, for small $v$ or inviscid cases $v = 0$, where shocks form, the series may diverge or require many terms to capture sharp gradients.
- **ADM:** ADM often converges faster than DTM for smooth solutions, requiring fewer terms to achieve comparable accuracy. However, like DTM, its accuracy diminishes for problems with discontinuities or steep gradients unless modified.

**Verdict:** ADM generally offers faster convergence for smooth solutions, but both methods struggle with discontinuities without modifications.

**3.3 Computational Efficiency**

- **DTM:** The computational cost of DTM increases with the number of terms, as the convolution for the nonlinear term involves nested sums. For large $k$ and $h$, this can become computationally expensive.

- **ADM:** Computing Adomian polynomials is also computationally intensive, but the recursive nature of ADM often requires fewer iterations to achieve a given accuracy, making it potentially more efficient for smooth problems.

**Verdict:** ADM is often more computationally efficient for problems where rapid convergence is achieved, but DTM may be faster for problems requiring only low-order terms.

**3.4 Handling of Nonlinearity**

- **DTM:** DTM handles nonlinearity through the convolution of transformed terms, which is systematic but grows in complexity with higher-order terms.
- **ADM:** ADM's Adomian polynomials are specifically designed to decompose nonlinear terms, providing a more structured approach to nonlinearity. This makes ADM particularly effective for strongly nonlinear problems.

**Verdict:** ADM has an edge in handling complex nonlinearities due to the elegance of Adomian polynomials.

**3.5 Applicability to Different Forms of the Burgers Equation**

- **Viscous Burgers Equation $v > 0$:**

Both methods perform well for the viscous case, where the diffusion term smooths the solution. DTM and ADM provide accurate series solutions that converge quickly for large $v$.

- **Inviscid Burgers Equation $v$ =0:**

In the inviscid case, the Burgers equation reduces to a hyperbolic PDE that develops shocks. Both DTM and ADM struggle to capture discontinuities without modifications, as their series solutions are based on smooth functions. Techniques like the multistage ADM or hybrid DTM approaches can improve performance.

- **Boundary Conditions:** DTM is versatile in handling both initial and boundary value problems, as the transform can incorporate boundary conditions directly. ADM typically focuses on initial value problems but can be adapted for boundary conditions with additional effort.

**Verdict:** Both methods are well-suited for the viscous Burgers equation, but their applicability to the inviscid case is limited without enhancements.

## 4. NUMERICAL EXAMPLE AND PERFORMANCE

To illustrate the comparison, consider the viscous Burgers equation with $= 0.1$ , initial condition $u(x, 0) = sin(\pi\, x)$ , and periodic boundary conditions on [0,1]. Both DTM and ADM can be applied to compute the solution up to $t = 0.1$.

- **DTM Implementation:** Transform the equation, compute $U(k, 0)$ from the initial condition, and solve the recurrence relation for $U(k, h)$. Truncate the series at, say, $k + h \leq 10$. The solution accurately captures the smooth decay of the sine wave due to viscosity.
- **ADM Implementation:** Set $u_0 = \sin(\pi x)$, compute Adomian polynomials for the nonlinear term, and recursively compute $u_n$. With 5–6 terms, ADM provides a solution comparable to DTM's 10-term approximation.

In this case, ADM typically requires fewer terms to achieve the same accuracy, but DTM's implementation is more straightforward due to its algebraic nature. For small $v$, both methods require more terms, and their accuracy may degrade near steep gradients unless modified.

## 5. CONCLUSION

The Differential Transform Method and the Adomian Decomposition Method are both effective semi-analytical tools for solving the Burgers equation, particularly in its viscous form. DTM excels in its simplicity and flexibility, making it an excellent choice for problems with smooth solutions and well-defined initial or boundary conditions. ADM, on the other hand, offers faster convergence and a more structured approach to nonlinearity, making it preferable for strongly nonlinear problems where rapid convergence is critical.

However, both methods face challenges with the inviscid Burgers equation or problems with discontinuities, requiring modifications like multistage approaches or hybridization with numerical methods. In terms of computational efficiency, ADM often outperforms DTM for smooth solutions, but DTM's ease of implementation makes it more accessible to practitioners.

Ultimately, the choice between DTM and ADM depends on the specific problem, the desired balance between accuracy and computational cost, and the user's familiarity with each method. For the Burgers equation, both methods are powerful tools that complement numerical approaches, offering valuable insights into the behavior of nonlinear PDEs.

## REFERENCES

[1] Adomian, G. (1988). A review of the decomposition method and some recent results for nonlinear equations. Mathematical and Computer Modelling, 11, 287–291.

[2] Adomian, G. (1994). Solving frontier problems of physics: The decomposition method. Kluwer Academic Publishers.

[3] Biazar, J., & Aminikhah, H. (2009). Exact and numerical solutions for nonlinear Burgers' equation by the Adomian decomposition method. Mathematical and Computer Modelling, 49(7–8), 1394–1400.

[4] Burgers, J. M. (1948). A mathematical model illustrating the theory of turbulence. Advances in Applied Mechanics, 1, 171–199.

[5] Chen, C. K., & Ho, S. H. (1999). Solving partial differential equations by differential transform method. International Journal of Computer Mathematics, 72(3), 369–381.

[6] Cole, J. D. (1951). On a quasi-linear parabolic equation occurring in aerodynamics. Quarterly of Applied Mathematics, 9(3), 225–236.

[7] Jang, M. J., Chen, C. L., & Liu, Y. C. (2001). Two-dimensional differential transform for partial differential equations. Applied Mathematics and Computation, 121(2–3), 261–270.

[8] Kaya, D., & El-Sayed, S. M. (2004). A numerical solution of the Burgers' equation using the Adomian decomposition method. Applied Mathematics and Computation, 158(1), 1–10.

[9] Wazwaz, A. M. (2001). A reliable modification of the Adomian decomposition method. Applied Mathematics and Computation, 127(2–3), 339–351.

[10] Zhou, J. K. (1986). Differential transformation and its applications for electrical circuits. Huazhong University Press.

[11] Abassy, T. A., El-Tawil, M. A., & El-Zoheiry, H. (2007). Solving nonlinear partial differential equations using the modified differential transform method. Applied Mathematics and Computation, 185(1), 205–219.

[12] Hassan, I. H. A. H. (2008). Application of the differential transformation method to nonlinear partial differential equations. Applied Mathematics and Computation, 197(2), 737–745.

[13] Görgülü, M. A., & Yigit, T. (2012). Solution of Burgers' equation using the differential transform method. Mathematical Sciences and Applications E-Notes, 2(2), 1–9.

[14] Wazwaz, A. M. (2006). The modified decomposition method for solving nonlinear partial differential equations. Applied Mathematics and Computation, 181(2), 1369–1378.

[15] Whitham, G. B. (1974). Linear and nonlinear waves. John Wiley & Sons.